Second-order differential equation with linear dependence in the solutions

Solve: $y'' - y = 2e^{x} + 3e^{-x} - \frac{3}{5}$

Solution

We solve the homogeneous part with the characteristic equation:

$$r^2 - 1 = 0$$

From here we find that the two roots are 1 and -1. Therefore, the homogeneous solution is:

$$y_H = C_1 e^x + C_2 e^{-x}$$

For the particular part, we propose a constant for $-\frac{3}{5}$:

$$y_C = A$$

We differentiate:

$$y_C' = 0$$

$$y_C'' = 0$$

We solve:

$$0 - A = -\frac{3}{5}$$

$$A = \frac{3}{5}$$

Therefore:

$$y_C = \frac{3}{5}$$

For the part involving $2e^x$, we propose $y_C = xJe^x$ since we need to break the linearity with the homogeneous solution. We differentiate:

$$y_C' = Je^x + xJe^x$$

$$y_C'' = Je^x + Je^x + xJe^x = 2Je^x + xJe^x$$

We substitute:

$$2Je^x + xJe^x - xJe^x = 2e^x$$

$$2J = 2$$

Therefore $J = 1$ and we have:

$$y_C = xe^x$$

For the part involving $3e^{-x}$, we propose $y_C = Kxe^{-x}$. We differentiate:

$$y_C' = Ke^{-x} - Kxe^{-x}$$

$$y_C'' = -Ke^{-x} - Ke^{-x} + Kxe^{-x} = -2Ke^{-x} + Kxe^{-x}$$

We solve:

$$-2Ke^{-x} + Kxe^{-x} - Kxe^{-x} = 3e^{-x}$$

$$-2K = 3$$

Therefore $K = -\frac{3}{2}$:

$$y_C = -\frac{3}{2}e^{-x}$$

The general solution is:

$$y_g = y_H + y_C = C_1 e^x + C_2 e^{-x} + \frac{3}{5} + xe^x - \frac{3}{2}e^{-x}$$